PMM U.S.S.R.,Vol.44,pp. 407-410

0021-8928/81/3 0407 $7.50/0

UDC 539.3

ON THE CONVERGENCE OF THE METHOD OF VARIABLE ELASTICITY PARAMETERS*

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Convergence of the method of variable elasticity parameters /1/ is proved for the boundary value problems of the deformational thermoplasticity of an anisotropic and inhomogeneous body. The proof of convergence of the method of elastic solutions is extended to the anisotropic case /2 – 7/ in order to provide a preliminary result, and the latter is used to deduce the existence and uniqueness of the solution of the boundary value problem in question.

It seems that until now no satisfactory proof of convergence of the method of variable elasticity parameters has been produced. The convergence of the method of elastic solutions for an isotropic homogeneous body has been proved in /5/. The inhomogeneous case was dealt with in /7/, and a more rapid version of the method was given in /3/.

We write the equations of state of a plastic, anisotropic and inhomogeneous material in the form

$$\sigma = \mathbf{D}\varepsilon - \mathbf{D}_0\varepsilon_0 - \sigma_0 \tag{1}$$

Here $\sigma, \varepsilon, \sigma_0, \varepsilon_0$ are six-dimensional vectors of stress, deformation, initial stresses and initial, e.g. thermal, structural and residual nonelastic deformations, determined in such a manner that the vector $\varepsilon = (\varepsilon_x, \varepsilon_y, \varepsilon_z, \varepsilon_{xy}, \varepsilon_{yz}, \varepsilon_{zx})$ corresponds to the symmetric second rank tensor ε_{ij} and $\mathbf{D}$ is a positive definite symmetric matrix dependent on the deformations and coinciding, in the case of an undeformed body, with the matrix $\mathbf{D}_0$ of elastic moduli.

Passing to the formulation of the boundary value problem, we define the operators $\mathbf{B}$ and $\mathbf{B}^*$ with the help of the following expressions:

$$\varepsilon = \mathbf{B}\mathbf{u} \Leftrightarrow \varepsilon_{ij} = {}^1/_2\,(\partial u_i / \partial x_j + \partial u_j / \partial x_i) \tag{2}$$

$$\mathbf{B}^*\sigma = \psi \Leftrightarrow \begin{cases} \partial\sigma_{ij} / \partial x_j = \psi_{1i}, & x \in \Omega / \partial\Omega \\ \sigma_{ij}\cos(n, x_j) = \psi_{2i}, & x \in \partial_2\Omega \end{cases} \tag{3}$$

Here $\mathbf{u}$ denote the displacements, $\psi = \{\psi_1, \psi_2\}$ is a pair of three-dimensional vector functions of the mass forces and surface loads defined, respectively, in a bounded regular domain Ω and on a fixed segment $\partial_2\Omega$ of its boundary $\partial\Omega$. The derivatives with respect to the coordinates in (2) and (3) should be regarded as classical or generalized, depending on the differential properties of $\mathbf{u}$ and σ. We note that the operator $\mathbf{B}^*$ is conjugate with $\mathbf{B}$ in a certain manner, since when $\mathbf{u} = 0$ on $\partial\Omega / \partial_2\Omega$, we have an identity

$$\int_\Omega \{\mathbf{B}\mathbf{u}, \sigma\}\, d\Omega = \int_\Omega (\mathbf{u}, \mathbf{B}^*\sigma)\, d\Omega + \int_{\partial_2\Omega} (\mathbf{u}, \mathbf{B}^*\sigma)\, dS$$

Here $(\ ,\ )$ denotes an Euclidean scalar product of three-dimensional vectors, and by the scalar product $\{\ ,\ \}$ of the stress (deformation) vectors we understand the contraction of the tensors corresponding to these vectors.

We assume that the displacements $\mathbf{u}$ are elements of the functional space U consisting of three-dimensional vector functions, square summable in Ω together with their first order partial derivatives, and equal to zero on $\partial_1\Omega = \partial\Omega / \partial_2\Omega$. The results of /8,9/ imply that the space U will be complete with respect to the norm induced by the scalar product

$$(\mathbf{u}, \mathbf{v})_\mathbf{E} = \int_\Omega \{\mathbf{B}\mathbf{u}, \mathbf{E}\mathbf{B}\mathbf{v}\}\, d\Omega \tag{4}$$

Here $\mathbf{E}$ is an arbitrary, positive definite symmetric matrix the components of which are bounded functions of the coordinates measurable in Ω. Combining the equations of state (1) with the geometrical (2) and static (3) conditions of the problem in question, we obtain the following equation for the displacements u:

$$\mathbf{K}\mathbf{u} = \mathbf{R}, \quad \mathbf{K} = \mathbf{B}^*\mathbf{D}\mathbf{B}, \quad \mathbf{R} = \mathbf{B}^*\mathbf{D}_0\varepsilon_0 + \mathbf{B}^*\sigma_0 + \psi \in \Psi, \quad \Psi = L^2(\Omega)^3 \times L^2(\partial_2\Omega)^3 \tag{5}$$

Here the operator $\mathbf{K}$ acts from the space U into the load space $\Psi \subset U^*$ conjugate to U. We note that $\mathbf{R} \in \Psi$ only when $\mathbf{D}_0\varepsilon_0$ and σ_0 have generalized first order derivatives square summable in Ω.

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*Prikl.Matem.Mekhan.,44,No.3,577-581,1980

In the case when D is bounded and independent of the deformations and hence of the displacements, while Ω satisfies the known conditions of regularity, equation (5) has, as was shown in /8,9/, a unique generalized solution. Under these conditions the operator K is elliptic, positive definite, and has a bounded inverse K^{-1}.

In the case when D is deformation dependent, the proof of existence of a solution of (5) is apparently known only for an iostropic material /6,10/. To show the existence and uniqueness of the solution of (5), we shall establish the convergence of the sequence of approximate solutions obtained with the help of the method of elastic solutions, and shall utilize the completeness of the space U.

Let us specify the properties of the matrix $\mathbf{D}$. We introduce, for the anisotropic body, the generalized spherical stress and deformation tensors σ^* and ε^*, generalized deviators $\mathbf{s}$ and $\mathbf{e}$, and their intensities τ and γ, using the relations

$$\sigma^* = \mathbf{N}^T\sigma,\quad \varepsilon^* = \mathbf{N}\varepsilon,\quad \mathbf{s} = \mathbf{J}^T\sigma,\quad \mathbf{e} = \mathbf{J}\varepsilon,\quad \tau = (m_0\,\{\mathbf{D}_0^{-1}\mathbf{s},\mathbf{s}\})^{1/2},\quad \gamma = (m_0^{-1}\,\{\mathbf{D}_0\,\mathbf{e},\mathbf{e}\})^{1/2} \tag{6}$$

Here $\mathbf{N}$ is a dimensionless matrix determined by the character of the anisotropy of the material and independent of the level of deformations, $\mathbf{J} = \mathbf{I} - \mathbf{N}$ where $\mathbf{I}$ is a unit matrix, $m_0 = \lim_{\gamma\to 0} d\tau/d\gamma$ and the index T denotes transposition.

The properties of the matrix N must ensure that the following relations hold:

$$\mathbf{N}^2 = \mathbf{N},\quad \mathbf{J}^2 = \mathbf{J},\quad \mathbf{NJ} = 0$$

Generalizing the equations of the deformation theory of plasticity to the anisotropic case, we assume that in the course of the deformation process σ^* and ε^* are connected by the linear relation

$$\sigma^* = \mathbf{D}_0\varepsilon^* = \mathbf{D}_0\mathbf{N}\varepsilon \tag{7}$$

and τ, γ are connected by a continuous relation unique for all types of the state of stress. In addition, the function $\tau(\gamma)$ has a piecewise continuous right derivative $m(\gamma)$ and the following inequality holds for almost all $\gamma \in [0,\infty]$:

$$0 < m_n \leqslant m(\gamma) \leqslant \tau(\gamma)/\gamma \leqslant m_0 < \infty \tag{8}$$

where m_n is a positive constant.

According to the assumption that a unique curve $\tau(\gamma)$ exists, from (1) and (6) it follows that the relation between the generalized deviators is given by the formula

$$\mathbf{s} = (m_0\gamma)^{-1}\tau(\gamma)\mathbf{D}_0\mathbf{J}\varepsilon \tag{9}$$

The relations (7), (9) coincide, for the corresponding choice of the matrix $\mathbf{N}$, with equations obtained in /11/ (see also /12/) and degenerate, in the case of an isotropic body, to the usual relations of the deformation theory of plasticity.

We write, in accordance with (7) and (9), the matrix D and operator $\mathbf{K}$ in the form

$$\mathbf{D} = \mathbf{D}_0 - \mathbf{D}_\Delta,\quad \mathbf{D}_\Delta = [1 - (m_0\gamma)^{-1}\tau(\gamma)]\mathbf{D}_0\mathbf{J},\quad \mathbf{K} = \mathbf{K}_0 - \mathbf{K}_\Delta,\quad \mathbf{K}_0 = \mathbf{B}^*\mathbf{D}_0\mathbf{B},\quad \mathbf{K}_\Delta = \mathbf{B}^*\mathbf{D}_\Delta\mathbf{B}$$

Since $\mathbf{D}_\Delta$ depends on the deformations, so does $\mathbf{K}_\Delta$, and it follows that they also depend on the displacements. As a result, equation (5) assumes the form

$$\mathbf{K}_0\mathbf{u} = \mathbf{K}_\Delta(\mathbf{u})\mathbf{u} + \mathbf{R} \tag{10}$$

We shall seek a solution of (10) in the form of a limit of the sequence $\{\mathbf{u}^n\}_{n=1}^{\infty}$ of approximate solutions constructed with the help of elastic solutions according to the formula

$$\mathbf{u}^{n+1} = \mathbf{K}_0^{-1}(\mathbf{K}_\Delta(\mathbf{u}^n)\mathbf{u}^n + \mathbf{R})\quad (n = 1, 2, \ldots, \infty)$$

By virtue of the completeness of the space U, the sufficient condition for the limit to exist is, that the mapping $\mathbf{A}: U \to U$ given by the operator $\mathbf{Av} = \mathbf{K}_0^{-1}\mathbf{K}_\Delta(\mathbf{v})\mathbf{v}$ be contracting, i.e. that

$$\exists\lambda < 1\quad \|\mathbf{Av}_1 - \mathbf{Av}_2\| \leqslant \lambda\|\mathbf{v}_1 - \mathbf{v}_2\|,\quad \forall \mathbf{v}_1, \mathbf{v}_2 \in U \tag{11}$$

To confirm that the property (11) holds, we estimate the norm of the (Fréchet) derivative $\mathbf{A}'(\mathbf{v})$ of the operator $\mathbf{A}$. According to the rule of differentiation of operators existing in the Banach space /13/, the derivative shown is a linear operator acting on the space U, and is defined by the relations

$$\mathbf{A}'(\mathbf{v})\mathbf{h} = \mathbf{K}_0^{-1}\mathbf{K}_\Delta(\mathbf{v})\mathbf{h} + \mathbf{K}_0^{-1}(\mathbf{K}_\Delta'(\mathbf{v})\mathbf{h})\mathbf{v},\quad \forall\mathbf{h} \in U,\quad (\mathbf{K}_\Delta'(\mathbf{v})\mathbf{h})\mathbf{v} = -(\gamma m_0)^{-2}\mathbf{B}^*(\tau(\gamma)/\gamma - d\tau/d\gamma)\{\mathbf{D}_0\mathbf{JBv}, \mathbf{JBh}\}\mathbf{D}_0\mathbf{JBv}$$

The operators $\mathbf{K}_0^{-1}\mathbf{K}_\Delta(\mathbf{v})$ and $\mathbf{K}_0^{-1}(\mathbf{K}_\Delta'(\mathbf{v})\cdot)\mathbf{v}$ are self-conjugate relative to the scalar product (4), provided that $\mathbf{E} = \mathbf{D}_0$. Since the norm of the self-conjugate linear operator $\mathbf{L}$ can be determined in the Hilbert space by the relation

$$\|\mathbf{L}\|_{\mathbf{E}} = \sup_{\mathbf{h}\in U}[|(\mathbf{Lh},\mathbf{h})_{\mathbf{E}}|(\mathbf{h},\mathbf{h})_{\mathbf{E}}^{-1}]$$

we have

$$\|\mathbf{A}'(\mathbf{v})\|_{\mathbf{D}_0} = \sup_{\mathbf{h}\in U}\left[\left|\int_\Omega (Q_1 + Q_2)\,d\Omega\right|\left(\int_\Omega \{\mathbf{Bh},\mathbf{D}_0\mathbf{Bh}\}\,d\Omega\right)^{-1}\right] \leqslant \sup_{\mathbf{h}\in U}\left[\left(\int_\Omega (|Q_1| + |Q_2|)\,d\Omega\right)\left(\int_\Omega \{\mathbf{Bh},\mathbf{D}_0\mathbf{Bh}\}\,d\Omega\right)^{-1}\right]$$

$$Q_1 = [1 - (m_0\gamma)^{-1}\tau(\gamma)]\{\mathbf{Bh}, \mathbf{D_0Bh}\}, \quad Q_2 = -(m_0\gamma)^{-2}(\tau/\gamma - d\tau/d\gamma)\{\mathbf{D_0JBv}, \mathbf{Bh}\}^2$$

Transforming Q_2 with the help of the Schwartz inequality and taking into account (6), (8), we have

$$\|\mathbf{A}'(\mathbf{v})\|_{D_0} \leqslant \sup_{\mathbf{h}\in U}\left[\left(\int_\Omega\left(1 - m_0^{-1}\frac{d\tau}{d\gamma}\right)\{\mathbf{Bh}, \mathbf{D_0JBh}\}\,d\Omega\right)\times\left(\int_\Omega\{\mathbf{Bh}, \mathbf{D_0Bh}\}d\Omega\right)^{-1}\right] \leqslant 1 - \inf_\Omega\left(m_0^{-1}\frac{d\tau(\gamma(\mathbf{v}))}{d\gamma}\right) \leqslant 1 - \frac{m_n}{m_0} \tag{12}$$

From the inequality (12) it follows that the method of elastic solutions converges independently of the initial approximation. Indeed, in accordance with the mean value theorem /13/ we have

$$\|\mathbf{Av}_2 - \mathbf{Av}_1\|_{D_0} \leqslant \operatorname*{vrai\,max}_{0\leqslant t\leqslant 1}\|\mathbf{A}'(t\mathbf{v}_2 + (1-t)\mathbf{v}_1)\|_{D_0} \times \|\mathbf{v}_2 - \mathbf{v}_1\|_{D_0} \leqslant (1 - m_n/m_0)\|\mathbf{v}_2 - \mathbf{v}_1\|_{D_0}$$

This means that condition (11) holds, and from this the convergence of the elastic solutions and the uniqueness of the solutions of (5) both follows. When $\mathbf{v} = \mathbf{u}$, the expression (12) yields an asymptotic estimate for the rate of convergence of the sequence $\mathbf{u}^n$ to $\mathbf{u}$, over the roots r. The rate of convergence is determined in accordance with /14/, as follows:

$$r = \lim_{n\to\infty}\sup\|\mathbf{u}^n - \mathbf{u}\|^{1/n} = \|\mathbf{A}'(\mathbf{u})\| \leqslant 1 - \inf_\Omega(m_0^{-1}d\tau(\gamma/\mathbf{v})/d\gamma) \tag{13}$$

The quantity r is chosen to characterize the rate of convergence, since, unlike the coefficient $q = \lim_{n\to\infty}\sup(\|\mathbf{u}^n - \mathbf{u}\|/\|\mathbf{u}^{n-1} - \mathbf{u}\|)$, it does not change when the space U is re-normed.

In accordance with the method of variable elasticity parameters, the solution $\mathbf{u}$ of (5), which represents a fixed point of the operator $\mathbf{C}: U \to U$ such that $\mathbf{C}(\mathbf{v}) = \mathbf{K}^{-1}(\mathbf{v})\mathbf{R}$, is sought in the form of a limit of the sequence $\{\mathbf{u}^n\}_{n=1}^\infty$ where $\mathbf{u}^n = \mathbf{K}^{-1}(\mathbf{u}^{n-1})\mathbf{R}$. The derivative $\mathbf{C}'(\mathbf{v})$ of the operator $\mathbf{C}(\mathbf{v})$ is given, at $\mathbf{v} = \mathbf{u}$, by the relation

$$\mathbf{C}'(\mathbf{u})\mathbf{h} = -\mathbf{K}^{-1}(\mathbf{u})(\mathbf{K}'(\mathbf{u})\mathbf{h})K^{-1}(\mathbf{u})\mathbf{R} = \mathbf{K}^{-1}(\mathbf{u})(\mathbf{K}_\Delta'(\mathbf{u})\mathbf{h})\mathbf{u}$$

By virtue of the properties of the solution $\mathbf{u}$ and the restrictions imposed on the function $\tau(\gamma)$, the matrix $\mathbf{D}$ corresponding to the distribution of deformations associated with the displacements $\mathbf{u}$, satisfies the conditions imposed on the matrix $\mathbf{E}$. It can therefore be used in constructing the scalar product $(,)_\mathbf{D}$ in U and of the norm $\|\cdot\|_\mathbf{D}$ equivalent to the initial norm $\|\cdot\|_{\mathbf{D}_0}$. Using the Schwartz inequality and the expressions (6), (8) and remembering that the matrix $\mathbf{D}$ can be represented by a sum of nonnegative matrices $\mathbf{D_0N}$ and $\tau(m_0\gamma)^{-1}\mathbf{D_0J}$, we have

$$\|\mathbf{C}'(\mathbf{u})\|_\mathbf{D} = \sup_{\mathbf{h}\in U}\frac{(\mathbf{K}^{-1}(\mathbf{K}_\Delta(\mathbf{u})\mathbf{h})u, \mathbf{h})_D}{(\mathbf{h}, \mathbf{h})_\mathbf{D}} \leqslant 1 - \inf_{\mathbf{h}\in U}\left[\left(\int_\Omega R_1\,d\Omega\right)\left(\int R_2\,d\Omega\right)^{-1}\right] \leqslant 1 - \operatorname*{vrai\,min}_\Omega\frac{d\tau/d\gamma}{\tau/\gamma} \tag{14}$$

$$R_1 = m_0^{-1}d\tau/d\gamma\,\{\mathbf{D_0JBh}, \mathbf{Bh}\}, \quad R_2 = \tau(m_0\gamma)^{-1}\{\mathbf{D_0JBh}, \mathbf{Bh}\}$$

The estimate (1.4) and the properties of the function $\tau(\gamma)$ together imply that $\|\mathbf{C}'(\mathbf{u})\|_\mathbf{D} < 1$, therefore $\mathbf{u}$ represents the point of attraction of the iterative process. This means that the method of variable elasticity parameters converges, provided that the initial approximation $\mathbf{u}^\circ$ is not too far from the solution $\mathbf{u}$ and the asymptotic estimate of the rate of convergence r is determined by the right hand side of the inequality (14).

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Translated by L.K.